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Injective mappings and solvable vector fields of Euclidean spaces ☆

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Abstract

We prove a sufficient condition for injectivity in a class of mappings defined on open connected subsets of the $\mathbb{R}^n$, for $n = 2$ and 3. The results relate solvability of appropriate vector fields with injectivity of the mapping.

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0. Introduction

Let Ω be an open connected subset of $\mathbb{R}^n$ and take $\Phi(\Omega)$ the subset of $C^\infty(\Omega, \mathbb{R}^n)$ consisting of the mappings F having invertible derivative $F'(x)$ for each $x \in \Omega$. A very general result for global injectivity is known in the Banach space setting, namely: If the image of F is simply connected, the Banach–Mazur theorem says that a necessary and sufficient condition for injectivity of F is that F be a proper mapping. (See Plastock [8].) In this paper we propose an alternative way, in some sense more analytic, to check the injectivity of F; we also give a sufficient condition weaker than a coercivity hypothesis, as the latter appears in the theory of elliptic partial differential operators. By coercivity hypothesis we mean that $\forall a \in \Omega$ there is a $C_a > 0$ such that $|F_a(x)| \geqslant C_a$ if $|x|$ is large, where $F_a(x) = F(x) - F(a)$ (= the translation of F in the source).

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In order to state our result we recall a concept of convexity used in the theory of linear partial differential operators on Ω. This concept is related to the solvability of such operators.

Definition 0.0. Let $P(x, D)$ be a smooth linear partial differential operator on Ω. We say that Ω is P-convex for supports if: For each compact set K of Ω there is a compact set K' of Ω such that if $u \in \mathcal{E}'(\Omega)$ with $\operatorname{supp}(P(x, D)^t u) \subset K$ then $\operatorname{supp}(u) \subset K'$.

Here, smooth means that the coefficients belong to $C^\infty(\Omega)$, $\mathcal{E}'(\Omega)$ is the space of compactly supported distributions on Ω, $\operatorname{supp}(u) =$ support of u and $P(x, D)^t$ is the formal transpose of $P(x, D)$. When P is a nonvanishing real vector field, Ω is P-convex for supports and a nontrapping condition for the trajectories of P on Ω is valid, it can be shown that $P : C^\infty(\Omega) \to C^\infty(\Omega)$ is surjective. (See Theorem 6.4.1 of Duistermaat and Hörmander [1].)

This generalized convexity condition will be shown to be essential for deciding when a locally injective mapping is globally injective.

If $F \in \Phi(\Omega)$, write $F = (f_1, \ldots, f_n)$; then for each $i \in \{1, 2, \ldots, n\}$ we denote $\mathcal{V}_{F,i}$ the C^∞ vector field defined by $\mathcal{V}_{F,i}(\phi)(x) = \det((F_{i,\phi})')(x)$, where $\phi \in C^\infty$ and the mapping $F_{i,\phi}$ is defined as follows: The jth-component of $F_{i,\phi}$ is equal to the jth-component of F if $j \neq i$ and is equal to ϕ if $j = i$. The connected components of $\{x \in \Omega;\ f_j(x) = c_j,\ j \neq i\}$, where $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \in \mathbb{R}$, define a C^∞ one-dimensional foliation of Ω. This foliation is precisely that of the trajectories of $\mathcal{V}_{F,i}$, because each f_j is a first integral of $\mathcal{V}_{F,i}$, if $j \in \{1, \ldots, n\}$ with $j \neq i$. For $n = 2$, $\mathcal{V}_{F,i} = H_{f_j}$, where $i \neq j$, where H_g is the Hamiltonian vector field associated to g. Then we introduce:

Definition 0.1. Let $F \in \Phi(\Omega)$. We say that Ω is F-convex if there is an open set Ω_1 of $\mathbb{R}^n$, $G_1 \in \Phi(\Omega_1)$ with $G_1(\Omega_1) = \Omega$ and $G_2 \in \Phi(F(\Omega))$, where each G_j is a diffeomorphism over its image such that: There are $n-1$ different indices $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$ such that Ω_1 is $\mathcal{V}_{F_1, i_j}$-convex in the sense of Definition 0.0 for $j \in \{1, \ldots, n-1\}$, where $F_1 = G_2 \circ F \circ G_1$.

For the two-dimensional case, our first result is:

Theorem 0.1. *Let Ω be an open connected subset of $\mathbb{R}^2$, and $F \in \Phi(\Omega)$, then the following hold*:

(i) *If Ω is F-convex then F is injective.*
(ii) *If F is injective and Ω is simply connected then Ω is F-convex.*

For the three-dimensional case we prove:

Theorem 0.2. *Let Ω be an open connected subset of $\mathbb{R}^3$ and $F \in \Phi(\Omega)$. If Ω is F-convex then F is injective.*

The next theorem proves that the notion of F-convexity imposes topological restrictions on Ω.

Theorem 0.3. *Let Ω be an open connected subset of $\mathbb{R}^2$ and $F \in \Psi(\Omega)$. If Ω is F-convex then Ω is simply connected, unless*

(i) *the complement Ω has exactly one nonempty compact connected component K and*
(ii) *every integral curve γ of $\mathcal{V}_{F_1,i_1}$ has either a nonempty ω-limit $\subset G_1^{-1}(K)$ or a nonempty α-limit $\subset G_1^{-1}(K)$, here $i_1 \in \{1,2\}$, F_1 and G_1 are as described in Definition* 0.1.

For the two-dimensional case the injectivity problem gained interest due to a counterexample found by Pinchuk (see [7]), which shows that locally injective polynomial mappings of the plane are not always globally injective. This type of condition also seems to be an important piece of information for deciding global stability for systems of ordinary differential equations. For example, consider the following:

Remark. Let $f, g \in C^1(\mathbb{R}^2)$ real valued functions. Consider the system of ordinary differential equations on the plane, given by

$$x'(t) = f\bigl(x(t), y(t)\bigr) \quad \text{and} \quad y'(t) = g\bigl(x(t), y(t)\bigr).$$

Assume that: $(f, g)(x, y) = (0, 0)$ if, and only if, $(x, y) = (0, 0)$ and that the system has no periodic trajectory. Moreover, assume also that the origin is a local attractor for the system. Then, the origin is a global attractor if, and only if, $\mathbb{R}^2 \setminus \{(0, 0)\}$ is $(f\partial_x + g\partial_y)$-convex for supports.

Proof the remark. Let Ω the basin of attraction of the equilibrium point. Clearly, by the Poincaré–Bendixson theorem, Ω is an unbounded simply connected open subset of $\mathbb{R}^2$. If $\Omega \neq \mathbb{R}^2$ then its boundary contains a trajectory α of the vector field (= trajectory of the system), with α unbounded at both ends. Take K a compact subset of $\mathbb{R}^2 \setminus \{(0, 0)\}$ given by the union of a simple closed continuous curve around the origin contained in Ω with the closure of a bounded open neighborhood of a point of α. It is easy to see that for such K there is no K'. Conversely, if $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$ we must have that $\mathbb{R}^2 \setminus \{(0, 0\}$ is $(f\partial_x + g\partial_y)$-convex, otherwise we would have a trajectory like α as above, concluding the proof of the remark. □

For a recent account on injectivity problems we will mention two references. The first one is a short paper by Gasull, Llibre and Sotomayor (see [3]), where the author learned how injectivity of mappings and global asymptotic stability for systems of ordinary differential equations in the plane are related. The second one is a paper by Meisters [5] which gives a fairly complete report of the subject.

As a consequence of Theorem 0.1 we prove the following quantitative result.

Corollary 0.1. *Let $\Omega = \mathbb{R}^2$ and $F \in \Phi(\mathbb{R}^2)$ as above. If for some $i \in \{1, 2\}$ there is a positive function $v \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $V(x) = v(x)H_{f_i}(x)$ is bounded away from 0 and the divergence of $V(x)$ is nonpositive everywhere, then $\mathbb{R}^2$ is F-convex and F is injective.*

The proof of the corollary is heavily based on the technique used by Olech (see [6]), which is basically 2-dimensional. It will be apparent in the proof why we restrict to the case $\Omega = \mathbb{R}^2$.

The organization of the paper is as follows. In Section 1 we prove a basic lemma and state a theorem which gives necessary and sufficient conditions in order for a real vector field to have a global transversal manifold; the latter can be found in [1] or [4]. Also we state a lemma which follows from two lemmas proved in Olech [6], and is a tool to prove Corollary 0.1. Also in this section, we prove a proposition which shows that we can separate, by a simple continuous curve, a compact connected component of the complement of an open connected subset of the plane from any other connected component. We also recall a lemma due to Fort (see [2]) on which the proof of the proposition is based. In Section 2 we prove Theorems 0.1 and 0.2. In Section 3 we prove Theorem 0.3 and Corollary 0.1 and an example is presented to show that the conclusion of Theorem 0.3 cannot be improved.

1. Preliminaries

First we prove a basic lemma.

Lemma 1.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let Σ be a 2-dimensional connected submanifold of Ω. Take $f \in C^\infty(\Omega, \mathbb{R})$, such that $\nabla f(p)$ is not normal to Σ at p, $\forall p \in \Sigma$. Assume that there is a $c \in \mathbb{R}$ such that $f^{-1}(\{c\}) \cap \Sigma$ has two different connected components. Then we can find a value c' of f and two different connected components Σ_1 and Σ_2 of $f^{-1}(\{c'\}) \cap \Sigma$ so that: For every compact set K contained in $\Sigma_1 \cup \Sigma_2$ and every $\varepsilon > 0$ there is a single connected component Σ'' of a level set of f such that the Euclidean distance between K and Σ'' is less than ε.*

We observe that if $n = 2$, the condition "∇f is not normal to Σ" means that "∇f does not vanish on Σ". Consider the one-dimensional foliation determined by the level curves of f. The result says that, on a neighborhood of the closure of a connected component of $\Omega \setminus (\Sigma_1 \cup \Sigma_2)$, the foliation is not globally linearizable, that is, the conclusion of the "Flow Box Theorem" does not hold globally for it, which is the main feature of the usual 2-dimensional Reeb foliation.

Proof of Lemma 1.1. Let Σ_1 and Σ_2 be the given connected components of $f^{-1}(\{c\}) \cap \Sigma$. We will keep the same notation even if we modify the choice of c. Clearly, there is no loss of generality to assume that there are $p_i \in \Sigma_i$ and a curve, $\gamma(t)$, $t \in [0, 1]$ contained in $\Omega \cap \Sigma$ and of class C^1, joining p_1 and p_2 such that $\Gamma \cap f^{-1}(\{c\}) = \{p_1, p_2\}$, where $\Gamma =$ Image of γ. Since at a point of Σ, $\nabla f(p)$ projects orthogonally onto a nonzero vector of the tangent plane to Σ at p, we can modify this curve in such way that the new curve also satisfies the following property: There is an integer k so that $\Gamma \cap f^{-1}(\{c'\})$ has at most

k points, for every c'. This is done by using the compactness of Γ. Namely, we first find a finite open cover $\{U_j\}$ of Γ such that each element of the cover is foliated by connected components of different level sets of f and $f^{-1}(\{c\}) \cap (\bigcup U_j) = (\Sigma_1 \cup \Sigma_2) \cap (\bigcup U_j)$. Using the flow of the gradient of f and matching curves on the intersection of two adjacent open sets of this cover we can modify γ so that it satisfies the required property.

With this γ we define $g(t) = f(\gamma(t))$; from the above construction $g'(t) \neq 0$ except at a finite set of points. From the Intermediate Value Theorem and the fact that $\Gamma \cap f^{-1}(\{c\}) = \{p_1, p_2\}$ we have that either of the following holds:

(i) $\operatorname{Min} g = c$, or
(ii) $\operatorname{Max} g = c$.

Furthermore, in any of these cases the corresponding extreme is attained only at the p_i's. There is no loss of generality in assuming that (i) holds. We consider the following situations:

Case (a). There is no local minimum for g in $(0, 1)$.

In this case the only other local extreme occurs at a point t_m of $(0, 1)$ and is a global maximum, because g is not a constant function. Since $g'(t) = 0$ on a finite set, then any value of g different from the maximum value occurs with multiplicity two on $[0, 1]$. Furthermore, near the maximum value, any value in $f(\Gamma)$, different from the maximum, occurs at the same connected component of a level set of f. Therefore taking $[t_1, t_2]$, with $g(t_1) = g(t_2)$, we obtain that the largest interval of $[0, 1]$ with $t_m \in (t_1, t_2)$, such that $f^{-1}(\{g(t)\}) \cap \Gamma$ with $t \in (t_1, t_2)$ is contained in a single connected component of the level set of f, for each $t \in (t_1, t_2)$.

We modify c if $t_1 > 0$, by taking $p_i = \gamma(t_i)$ and $c = f(\gamma(t_i))$ with $i \in \{1, 2\}$. In any case this gives that for any value in $g((t_1, t_2))$ its pre-image by f in Γ is contained in a single connected component of level set of f. Consequently, given K and ε as in the statement of the lemma it is enough to take $c' = g(t)$ with $t > t_1$ near to t_1.

Case (b). There is a local minimum for g in $(0, 1)$.

Starting from a point $\gamma(t_m)$ where $g(t_m)$ is global maximum value for $g(t)$, since it is an isolated critical point, we can as in case (a) define $[t_1, t_2]$, satisfying the conditions described above. Now $t_1 > 0$, but making the change of c as in case (a), we derive the same conclusion.

This proves Lemma 1.1. □

Our next goal is to state a lemma which is a consequence of Lemmas 1 and 2 of Olech, and we refer to [5] for a proof. Let $F^{\perp} = (-f_2, f_1)$ and take two points x^1 and x^2 in an orbit of $x'(t) = F(x(t))$ and y^1 and y^2 in a nearby orbit of the same system such that: x^j is connected to y^j by an orbit α_j of $w'(t) = F^{\perp}(w(t))$ for $j = 1, 2$. Here x^2 is connected to x^1 by one such orbit starting at x^1 and going forward in time, and analogously for y^2 and y^1. For nearby points we have that those segments of orbits give the boundary of a bounded simply connected domain.

Lemma 1.2. *Under the notation above, if in addition F has nonpositive divergence in Ω then $D(x^2, y^2) \leqslant D(x^1, y^1)$.*

Here $D(x^j, y^j) = \int_0^{s_0} \|F(\alpha_j(t(s)))\| |ds|$, where $[x_j, y_j]$ is the segment defined by α_j, $|ds|$ is the line element and s_0 the length of the segment $[x_j, y_j]$.

Finally, we recall the following result which gives necessary and sufficient conditions for a C^∞ linear real vector field be a surjective operator on $\mathcal{C}^\infty(\Omega)$. We refer to Duistermaat and Hörmander [1, Theorem 6.4.1] for a complete account, and also Hörmander [3, Lemma 26.1.11]; here we only report the two equivalent properties of those results that we will use in this paper.

Theorem 1.1. *Let M be a C^∞ manifold and L a C^∞ vector field on M. Then the following conditions on L are equivalent*:

(a) *No complete integral curve of L is relatively compact, and for every compact set K in M there is another compact K' in M containing every compact interval on an integral curve with end points in K.*
(b) *There exists a manifold M_0, and an open neighborhood M_1 of $M_0 \times \{0\}$ which is convex in the $\mathbb{R}$ direction, and a diffeomorphism $M \to M_1$ which carries L into the operator ∂_t if points in $M_0 \times \mathbb{R}$ are denoted by (y_0, t).*

Now we consider the following question: Let Ω be an open connected subset of the plane. Assume that K_0 is a nonempty compact connected component of $\mathbb{R}^2 \setminus \Omega$. Let K_1 be any other connected component of $\mathbb{R}^2 \setminus \Omega$: Can we separate K_0 from K_1 by a simple continuous curve contained in Ω?

Using the stereographic projection we identify Ω with an open connected subset $\widetilde{\Omega}$ of the sphere S^2, identifying ∞ with north pole. Then $K = S^2 \setminus \widetilde{\Omega}$ is a closed subset of the sphere, and we write K as disjoint union of its connected components $\bigcup_{j\in J} K_j$; here we can consider $J \subset [0,1]$. Now we show that if $j \in J$ is such that $\operatorname{Inf}_{j\neq k}\{d(K_j, \bigcup_{k\in J,k\neq j} K_k)\} > 0$ then there is a C^∞ simple closed continuous curve Γ contained in $\widetilde{\Omega}$ such that K_j is contained in one connected component of $S^2 \setminus \Gamma$ and $\bigcup_{k\in J,k\neq j} K_k$ is contained in the other connected component of $S^2 \setminus \Gamma$. This can be seen by first using the Whitney extension theorem to find a real valued function $\phi \in C^\infty(S^2)$ such that $0 \leqslant \phi(x) \leqslant 1$ $\forall x$, $\phi^{-1}(\{0\}) = K_j$ and $\phi^{-1}(\{1\}) = \bigcup_{k\in J,k\neq j} K_k$, since the latter is also a compact subset of S^2. Then by Sard's theorem there is a regular value λ of ϕ, which we can choose so that $\phi(\infty) \neq \lambda$. Finally we take $\Gamma =$ a connected component of $\phi^{-1}(\{\lambda\})$. Therefore the only thing left to prove is that Γ is compact, but this follows from the hypothesis of λ is a regular value of a C^1 function defined on a compact surface without boundary (in our case S^2), by the Implicit Function Theorem.

The argument above does not applies when $\operatorname{Inf}_{j\neq k}\{d(K_j, \bigcup_{k\in J,k\neq j} K_k)\} = 0$. As first step to answer the posed question we state the following lemma which is consequence of a lemma due to Fort, see [2].

Lemma 1.3. *Let Ψ be a connected open set of S^n, and let p and q be points in some connected component K of the boundary $\partial\Psi$ of Ψ. Corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that if $p' \in B_\delta(p) \cap \Psi$ and $q' \in B_\delta(q) \cap \Psi$, then there is a continuous arc α having end points p' and q' such that $\alpha \subset T(\varepsilon, \partial\Psi) \cap \Psi$.*

Here $B_\xi(r)$ is the open ball centered at r and radius ξ, and $T(\xi, X) = \{r \in S^n;\ d(r, X) < \xi\}$ = open tubular neighborhood of X, and the metric is that induced on S^n by the Euclidean metric of $\mathbb{R}^{n+1}$.

Now we present a complete proof of the topological result which gives a positive answer to the question. Even though the result is elementary and seems intuitively obvious, as far as we know the only situation for which this proposition is known for $\mathrm{Inf}_{j\neq k}\{d(K_j, \bigcup_{k\in J, k\neq j} K_k)\} = 0$ is when $\mathbb{R}^2 \setminus \Omega$ is totally disconnected, and the proof follows from Dimension Theory.

Proposition 1.1. *Let Ω be a connected open subset of $\mathbb{R}^2$ and assume that K and L are distinct connected components of $\mathbb{R}^2 \setminus \Omega$, with K compact. Then there is a continuous simple closed curve Γ contained in Ω such that K is contained in one component of $\mathbb{R}^2 \setminus \Gamma$ and L is contained in the other component of it.*

Proof. From the hypothesis of K being bounded, using stereographic projection, $\mathrm{sp} : \mathbb{R}^2 \cup \{\infty\} \to S^2$, we reduce the assertion to an open connected subset $\mathrm{sp}(\Omega) = \widetilde{\Omega}$ of S^2 and the corresponding $\mathrm{sp}(K) = \widetilde{K}$, with $\mathrm{sp}(\infty) \notin \widetilde{K}$, which is different from $\mathrm{sp}(L) = \widetilde{L}$. If

$$d\big(\widetilde{K}, S^2 \setminus \big(\widetilde{\Omega} \cup \widetilde{K}\big)\big) > 0 \tag{1.1}$$

the proof follows from the arguments given before. So we may assume that (1.1) is not true.

As above, the first step of the proof consists in constructing a simple closed continuous curve Δ in S^2, not passing through the north pole, such that $\widetilde{K}$ is contained in one connected component of $S^2 \setminus \Delta$ and $\widetilde{L}$ is contained in the other connected component of $S^2 \setminus \Delta$. If such a curve is contained in $\widetilde{\Omega}$ we take $\Gamma = \mathrm{sp}^{-1} \Delta$, and the proof is finished. Therefore we may assume that $(\widetilde{K} \cup \widetilde{L}) \cap \Delta = \emptyset \neq \Delta \cap (S^2 \setminus (\widetilde{\Omega} \cup \widetilde{K} \cup \widetilde{L}))$.

The second step starts by defining $J' = \{j \in J;\ \widetilde{K}_j \cap \Delta \neq \emptyset\}$, which is a nonempty subset of J by the first step. At this point we recall that, from the fact that S^2 with the usual metric is a compact metric space, the space of closed subsets of S^2, namely $\mathcal{C}_{S^2} = \{C \subset S^2;\ \overline{C} = C\}$, with the Hausdorff metric is compact, hence sequentially compact. Therefore, arguing by contradiction, from Lemma 1.3, we have that

$$\varepsilon_0 = d\Bigg(\bigcup_{j\in J'} \widetilde{K}_j, \widetilde{K} \cup \widetilde{L}\Bigg) > 0 \tag{1.2}$$

for otherwise we could find a sequence of continuous curves joining Δ to a point arbitrarily closed to, say, $\widetilde{K}$, so that it is uniformly convergent by the sequential compactness of $\mathcal{C}_{S^2}$. The limit curve will be contained in $S^2 \setminus \widetilde{\Omega}$. By connectivity, this would imply that $\widetilde{K}$ intersects Δ, which is a contradiction.

The third step consists in using of Lemma 1.3 in order to deform Δ to obtain a curve Γ satisfying the required properties; this is done in the sequel: First take $\Delta_0 = (\bigcup_{j\in J'} \widetilde{K}_j) \cap \Delta$ which is a nonempty closed subset of Δ. From the hypothesis of $\widetilde{\Omega}$ being a connected subset not contained in either side of Δ we have that $\Delta \setminus \Delta_0 \neq \emptyset$. Now, we will define an open cover, for Δ_0 in the induced topology in Δ. Take a fixed orientation of Δ. Let $r \in \Delta_0$ and the largest arc $[L(r), R(r)]$ contained on Δ_0, which is well defined by the orientation given in Δ. In particular we have $r \in [L(r), R(r)]$ and $\{L(r), R(r)\} \subset \partial \widetilde{K}_j$ for some $j \in J'$, by connectivity of $\widetilde{K}_i$, $i \in J$. So we apply Lemma 1.3 to $\widetilde{K}_j$, taking $\varepsilon = \varepsilon_0/3$. We obtain δ_r, so that if $p', q' \in \widetilde{\Omega} \cap \Delta \cap T(\delta_r, \partial\widetilde{K}_j)$ then there is a continuous curve γ_r joining p' to q' so that $\gamma_r \subset T(\varepsilon_0/3, \partial\widetilde{\Omega})$. Clearly the distance $d(\gamma_r, \widetilde{K} \cup \widetilde{L}) \geqslant 2\varepsilon_0/3$. Now using the connectivity of $\widetilde{\Omega}$ and the same technique used on step 2, we can take δ_r sufficiently small so that

$$\gamma_r \subset T\big(\varepsilon_0/3, \partial\widetilde{K}_j\big) \cap \widetilde{\Omega} \tag{1.3}$$

for any two points $p', q' \in \widetilde{\Omega} \cap \Delta \cap T(\delta_r, \partial\widetilde{K}_j)$.

Also, we observe that from the connectedness hypothesis on the involved sets such pair a p', q' exists arbitrarily near $L(r)$ and $R(r)$, respectively, so that the open arc (p', q') contains $[L(r), R(r)]$. By modifying p' and q', if necessary, we can assume that $\widetilde{\Omega} \cap$(arc of Δ with endpoints p' and $L(r)) \subset B_{\delta_r}(L(r))$ and $\widetilde{\Omega} \cap$(arc of Δ with endpoints $R(r)$ and $p') \subset B_{\delta_r}(R(r))$.

Call $p_r = p'$ and $q_r = q'$, for some choice of p' and q' as above. We consider the open cover of Δ_0 given by $\{\mathrm{arc}(p_r, q_r)\}_{r \in \Delta_0}$. By compactness of Δ_0 we have a finite subcover, $\mathrm{arc}(p_{r_1}, q_{r_1}), \ldots, \mathrm{arc}(p_{r_k}, q_{r_k})$ of Δ_0. Now fix p_{r_1} and rename the remaining p_{r_j} by saying that $p_{r_l} < p_{r_{l+1}}$ if the $\mathrm{arc}(p_{r_1}, p_{r_l}) \subset \mathrm{arc}(p_{r_1}, p_{r_{l+1}})$. We have two situations:

The first one is when two consecutive arcs intersect; in this case, $q_{r_i} \in \mathrm{arc}(p_{r_{i+1}}, L(r_{i+1}))$, for $i \in \{1, \ldots, l-1\}$ (or $q_{r_k} \in \mathrm{arc}(p_{r_1}, L(r_1))$ if $l = k$). From the properties of the p_r's and q_r's we can apply Lemma 1.3 once again and connect q_{r_i} to $p_{r_{i+1}}$ by a continuous curve β_{r_i} (or connect q_{r_k} to p_{r_1} by a continuous curve β_{r_k}), contained in $T(\varepsilon_0/3, \partial\widetilde{K}_i) \cap \widetilde{\Omega}$.

The second situation is when two consecutive arcs do not intersect, in this case we take β_{r_i} to be the piece of arc of $\Delta \cap \widetilde{\Omega}$ joining the respective q_{r_l} to $p_{r_{l+1}}$. Therefore the closed simple continuous curve Γ obtained by connecting γ_{r_l} to $\beta_{r_{l'}}$ satisfies the required property. □

Example. Let $\Omega = \mathbb{R}^2 \setminus F$, where $F = \bigcup_{n=0}^{\infty} F_n$, $F_0 = \{(-1, y);\ \ y \geqslant 0\}$, $F_1 = \{(1, y);\ y \geqslant 0\}$ and $F_n = \{(x, -n(x+1-1/n)(x-1+1/n));\ -1+1/n \leqslant x \leqslant 1-1/n\}$, for $n \geqslant 2$. By taking $K = F_0$ and $L = F_1$, this example shows that the compactness' hypothesis on K cannot be omitted in Proposition 1, even if we allow the curve to be open, continuous, with endpoints at ∞.

2. Proofs of Theorems 0.1 and 0.2

Proof of Theorem 0.1. First we will prove (i). We can assume that the G_j's are equal to the identity mapping. After rearrangement we can assume that f_1 is a component of F which give us that Ω is F-convex.

Assume that F is not injective, so that there exist $p_1, p_2 \in \Omega$ such that $F(p_1) = F(p_2)$, with $p_1 \neq p_2$. Furthermore, since $H_{f_1}(f_2) = \det(F')$ never vanishes we have that p_i, $i = 1$ and 2, belongs to different connected components of $f_1^{-1}\{f_1(p_1)\}$. Since a connected component of a level set of f_1 is a characteristic curve of H_{f_1} we have by Lemma 1 that Ω cannot be H_{f_1}-convex, which is a contradiction.

To prove (ii) we can assume that Ω is different from the entire plane and $F = I$. Now applying the Riemann mapping theorem by composing with the Riemann mapping and its inverse we reduce to the case that $\Omega =$ open ball, implying that Ω is F-convex.

This ends the proof of the Theorem 0.1. □

Proof of Theorem 0.2. We can assume that the G_j's are equal to the identity mapping. After rearrangement, we can assume that the indices giving the F-convexity of Ω are $i_j = j$ for $j \in \{1, 2\}$. Assume that F is not injective and take $p_1, p_2 \in \Omega$ such that $F(p_1) = F(p_2)$ but $p_1 \neq p_2$. Take $\gamma_{p_i,j}$ be the complete trajectory of $\mathcal{V}_{F,j}$ passing through p_i.

We will divide our proof into 4 steps, the first one is solely a consequence of $F \in \Phi(\Omega)$:

Step 1.

(a) p_1 and p_2 belong to different complete integral curves of $\mathcal{V}_{F,j}$, for $j \in \{1, 2\}$.
(b) $\mathcal{V}_{F,j}$ does not have any complete integral curve inside a compact subset, for any j.

Proof. (a) First notice that $\mathcal{V}_{F,1}(f_1) = \det(F') = -\mathcal{V}_{F,2}(f_2)$ is never zero from the hypothesis of $F \in \Phi(\Omega)$, therefore f_j is strictly increasing (or decreasing) along the trajectories of $\mathcal{V}_{F,j}$. So $f_j(p_1)$ cannot be equal to $f_j(p_2)$ if p_1 and p_2 belongs to the same complete integral curve of $\mathcal{V}_{F,j}$. Then (a) follows.

(b) Assume that ω-limit (or the α-limit) of $\gamma_{p,j}$ is a nonempty set, for some j. Without loss of generality we can assume that there is a point $q \in \Omega$ in the ω-limit of $\gamma_{p,j}$. But from the description of $\gamma_{q,j}$, namely it is a connected component of $\{r;\ f_k(r) = f_k(q),\ \forall k \neq j\}$, we have that $\gamma_{q,j} = \gamma_{p,j}$, from (a). So $\gamma_{p,j}$ is periodic, but this contradicts the fact that f_j is strictly increasing (or decreasing) on $\gamma_{p,j}$. □

Step 2. Each of the vector fields $\mathcal{V}_{F,j}$, with $j \in \{1, 2\}$, satisfies condition (a) of Theorem 1.1.

Proof. This follows from step 1(b) and the hypothesis. □

Step 3. We cannot have both p_i, $i \in \{1, 2\}$, belonging to the same connected component of

$$\bigl\{q;\ f_j(q) = f_j(p_1),\ \forall j \notin \{1, 2\}\bigr\}.$$

Proof. Assume that the result is false, then we consider Σ the connected component of $\{q;\ f_j(q) = f_j(p_1),\ \forall j \notin \{1, 2\}\}$, so that $p_i \in \Sigma$ $\forall i$. From step 2 and Theorem 1.1 we have that (b) of Theorem 1.1 holds. Take $f = f_1$ and $c = f_1(p_1)$; from the hypothesis of $F \in \Phi(\Omega)$ we have that the hypotheses of Lemma 1.1 hold for the data Σ, f_1 and c. Let Σ_i, $i \in \{1, 2\}$, obtained by Lemma 1.1, then each Σ_i is a subset of the same level set of f_1. Take $q_i \in \Sigma_i$; once again by Lemma 1.1, near q_1, there is a single connected component of a level set of f_1 so there its intersection with Σ passes arbitrarily close to q_2. This implies that a transversal manifold of $\mathcal{V}_{F,2}$, which passes through q_1 and q_2 intersects twice the same orbit of this vector field, in fact one which is contained in Σ. Contradicting item (b) of Theorem 1.1 is not valid. Proving therefore that step 3 is true. □

Step 4. Let $n = 3$; then we cannot have the p_i, $i \in \{1, 2\}$, belonging to different connected components of $\Sigma_{1,2} = \{q;\ f_j(q) = f_j(p_1),\ \forall j \notin \{1, 2\}\}$.

Proof. From step 2 we have that $\mathcal{V}_{F,j}$ has a global transversal, call it $\mathcal{T}_j$, respectively, for $j \in \{1,2\}$, satisfying the properties of Theorem 1.1(b). Step 4 is a consequence of the following properties:

(i) From the description of $\gamma_{q,j}$ and the properties of $\mathcal{T}_j$ we have that $\Sigma_{1,2}$ is transversal to $\mathcal{T}_j$, for $j \in \{1,2\}$. Furthermore, the intersection of $\Sigma_{1,2}$ with $\mathcal{T}_j$ contains a nonconnected smooth curve, because locally it is a smooth curve and the nonconnectivity follows from the fact that p_i ($i = 1$ and 2) belongs to different connected components of $\Sigma_{1,2}$.

(ii) Call $\Sigma_{1,2}^{p_1}$ and $\Sigma_{1,2}^{p_2}$ the connected components of $\Sigma_{1,2}$ that contains p_1 and p_2, respectively. (Observe that in the 3-dimensional case those surfaces are different connected components of $f_3^{-1}(\{f_3(p_1)\})$.) Consider the configuration $\mathcal{T}_1$, f_3 and $c = f_3(p_1)$. We observe that the level surfaces of f_3 cannot be tangent to $\mathcal{T}_1$, by (i).

From the last observation above we can apply Lemma 1.1 to this configuration. Consider $c'(= f_3(p_1') = f_3(p_2'))$ and disjoint connected smooth curves $\alpha_{p_1'} = f_3^{-1}(\{c'\}) \cap \mathcal{T}_1$ and $\alpha_{p_2'} = f_3^{-1}(\{c'\}) \cap \mathcal{T}_1$, with $p_1', p_2' \in \mathcal{T}_1$ determined by the conclusion of the lemma.

(iii) With the notation of (i) and (ii) we consider the trajectories of the flow of $\mathcal{V}_{F,2}$ starting at p_j', $j \in \{1,2\}$. Such trajectories are contained in different connected components, say Σ_j', of $f_3^{-1}(\{c'\})$, for $j \in \{1,2\}$, because otherwise the proof of step 3 yields a contradiction. On the other hand, applying Lemma 1.1 to the configuration $\Sigma_{1,2}$, f_2 and $c = f_2(p_1')$, since the trajectories of $\mathcal{V}_{F,2}$ (contained in $\Sigma_{1,2}$) are transversal to $\mathcal{T}_2$, there is a connected component of a level set of f_3 intersecting $\mathcal{T}_2$ at two points. But this is in contradiction with the fact that $\mathcal{T}_2$ is a global transversal of $\mathcal{V}_{F,2}$. $\square$

But steps 3 and 4 contradicts each other, so the reduction to absurd is completed. This ends the proof of Theorem 0.3. $\square$

Example. For $n = 2 + j$, with $j \geqslant 1$, consider $F(x,y,z) = (e^x \cos(y), e^x \operatorname{sen}(y), z_1, \ldots, z_j)$. It is easy to check that $F \in \Phi(\mathbb{R}^n)$ is not globally injective on $\mathbb{R}^n$. But $\mathbb{R}^n$ is $\mathcal{V}_{F,k+2}$-convex for $1 \leqslant k \leqslant j$. Therefore, for $n \geqslant 3$ the existence of $n-2$ globally solvable vector fields of the above type is not sufficient to guarantee that F is injective.

3. Proofs of Theorem 0.3 and Corollary 0.1

Proof of Theorem 0.3. We prove it by contradiction. There is no loss of generality by assuming that $\Omega_1 = \Omega$ and $i = 1$. We use the same notation as in the proof of Proposition 1.1. By hypothesis of contradiction we have that Ω is not simply connected therefore $J' = \{j \in J;\ K_j$ is a nonempty compact connected component of $\mathbb{R}^2 \setminus \Omega\} \neq \emptyset$.

Firstly we will show that there is $j \in J'$ and a trajectory δ of H_{f_1} such that

$$\bigl(\omega(\delta) \cup \bigl(\alpha(\delta)\bigr)\bigr) \cap K_j \neq \emptyset. \tag{3.1}$$

Suppose this is not the case. Fixed $j \in J'$. If $\mathbb{R}^2 \setminus \Omega$ has more than one connected component apply Proposition 1.1 for $K = K_j$ and L any other connected component of $\mathbb{R}^2 \setminus \Omega$, to find a C^1 simple curve $\gamma(t)$ in Ω, such that: It has nonzero tangent vector everywhere and contains K_j in its interior. If $\Omega = \mathbb{R}^2 \setminus K_j$ the existence of such curve γ follows easily. As in the proof of Lemma 1.1, since ∇f_1 never vanishes, we can deform the initial γ so that for the new γ we have that $f_1(\gamma_i(t))$ has a finite number of critical points. Consider $K = \text{Image}(\gamma)$ and the associated K' as in Definition 0.1. Take $\Omega_i = \text{Inside}(\text{Image}(\gamma)) \cap \Omega$; here the inside of Image(γ) exists in $\mathbb{R}^2$ by the Jordan Curve Theorem. Note that K cannot be contained in a single connected component of a level set of f_1, because as before this would imply that H_{f_1} has a periodic trajectory, but from the positivity(or negativity) of $H_{f_1}(f_2) = \det(F')$ then f_2 is strictly increasing (or decreasing) along this periodic trajectory of H_{f_1}, which is impossible.

Take $M = \text{Max}_K\, f_1$, the maximum value of f_1 on K, let p_0 be a point of K such that $f_1(p_0) = M$ and let Δ_0 be the connected component of the level set $\{p;\ f_1(p) = M\}$ that passes through p_0. Above we proved that Δ_0 cannot be a subset of K. Furthermore, from the hypothesis, Δ_0 must be contained on $\Omega \setminus \Omega_i$.

We parameterize Δ_0 as a trajectory of H_{f_1} (or $-H_{f_1}$), call it $\delta_0(t)$, so that if $(a_{\delta_0}, b_{\delta_0})$ is the maximal domain of definition of δ_0 we have: For some $s_i^{\delta_0}$, $i \in \{1,2\}$, with $a_{\delta_0} < s_1^{\delta_0} \leqslant s_2^{\delta_0} < b_{\delta_0}$ we have $\delta_0(s_i^{\delta_0}) \in K$ and $\delta_0(t) \notin \Omega_1 \cup K$ if $t < s_1^{\delta_0}$ or $t > s_2^{\delta_0}$, since from the hypothesis we have that $(\omega(\delta) \cup \alpha(\delta)) \cap K_{j'} = \emptyset$, $\forall K_{j'} \subset \Omega_i$ and δ.

We write Ω as disjoint union of two open connected subsets named $\Omega^+_{\Delta_0}$ and $\Omega^-_{\Delta_0}$, and the closed subset Δ_0. We call $\Omega^+_{\Delta_0}$ the one not intersecting Ω_i.

From a continuous dependence's argument it follows that $K' \cap \Omega_i \neq \emptyset$. To see this note that, from the hypothesis that ∇f_1 never vanishes, near p_0 there is an unique connected component Δ_c of the level curve of f_1, corresponding to a value $M - c$, for $c > 0$ small. For such c's, Δ_c satisfies:

(1) The trajectory $\delta_c(t)$ of H_{f_1} corresponding to the connected component Δ_c, of this level set, has α-limit and ω-limit contained in the part of the boundary of Ω outside K;
(2) If $(a_{\delta_c}, b_{\delta_c})$ is the maximal domain of definition of δ_c, we can take $a_{\delta_c} < s_1^{\delta_c} \leqslant s_2^{\delta_c} < b_{\delta_c}$, where $s_1^{\delta_c}$ and $s_2^{\delta_c}$ are, respectively, the largest and smallest t, such that if $t \in (a_{\delta_c}, s_1^{\delta_c}) \cup (s_2^{\delta_c}, b^{\delta_c})$ then $\delta_c(t) \notin \Omega_1 \cup K$;
(3) Ω can be written as the disjoint union of the closed subset Δ_c and two open connected subsets, named $\Omega^+_{\Delta_c}$ and $\Omega^-_{\Delta_c}$, with $\Omega^+_{\Delta_0} \subset \Omega^+_{\Delta_c}$ and $\Omega^+_{\Delta_c} \cap \Omega_1 \neq \emptyset$;
(4) In fact, the connected part Δ_c^0 of Δ_c joining $\delta_c(s_1^{\delta_c})$ to $\delta_c(s_2^{\delta_c})$ is contained in K' and has nonempty intersection with Ω_1; and
(5) $\Omega_c^+ \subset \Omega_{c'}^+$ if $c' > c$.

Given a $c_0 \in (0, M - \text{Min}_{p\in K}\{f_1(p)\})$ satisfying the conditions (1) to (5) we can make a similar construction by replacing $c = 0$ by $c = c_0$. From this we have that:

$$I = \bigl\{c > 0,\ \forall d \in (0,c),\ \Delta_d \text{ and } \delta_d \text{ constructed as above satisfy (1)–(5)}\bigr\}$$

is a open bounded interval, in fact bounded from above by $M - \text{Min}_{p\in K}\{f_1(p)\}$. Furthermore, defining

$$\Omega_0' = \bigcup_{c\in I} \Delta_c^0.$$

We have that $\Omega_0' \subset K'$.

Clearly, (3.1) holds if we observe that the Euclidean distance, $D = \mathrm{dist}(K', \partial\Omega)$, from K' to the boundary of Ω, is equal to zero, since this will contradicts that K' is a compact subset of Ω.

We will show that $D = 0$ follows from the construction (1)–(5), determining a finite number of sets of the form Ω_0', so that its union has distance zero to the boundary of Ω. If the closure of the above Ω_0' is a noncompact subset of Ω then $D = 0$, so let us assume the contrary: We take $c_M = \sup\{c \in I\}$ and then consider the subset of the level set $f_1^{-1}(\{c_M\})$ which is obtained as the limit, over compact sets, of Δ_c as $c \to c_M$; this subset is nonempty by compactness of K. Furthermore, consider a connected component Δ_{c_M} of such a level set which intersects K; similarly to the case $c = 0$, considering Δ_{c_M} as a trajectory of H_{f_1}, we analyze the following possibilities for its ω-limit, and α-limit sets: (a) the ω-limit (or α-limit) set has a point on the part of the boundary of Ω which is inside K, and (b) both are contained in the part of the boundary of Ω which is outside K. Let us analyze those two cases:

Case (a) By the continuous dependence argument any $q \in \Delta_{c_M} \cap \Omega_0$ is the limit of a sequence of points $\in \Omega_0'$, so $\mathrm{dist}(\Omega_0', \partial\Omega) = 0$, hence $D = 0$.

Case (b) For each point $p \in K$ consider the trajectory, $\delta_{f_1(p)}$, of H_{f_1} which passes through p. So either $\omega_{\delta_{f_1(p)}} \cup \alpha_{\delta_{f_1(p)}}$ has a point on the part of the boundary of Ω which is inside K or $\omega_{\delta_{f_1(p)}} \cup \alpha_{\delta_{f_1(p)}}$ is a subset of the part of the boundary of Ω which is outside K. The first situation cannot happen, because as in case (a) by compactness of K, we would find a point $q \in K$ such that any point, of the nonempty set, $(\omega_{\delta_{f_1(q)}} \cup \alpha_{\delta_{f_1(q)}}) \cap \Omega_0$ is limit of points of K', proving that $D = 0$. Finally the last situation; from what we have just proved and by compactness of K implementing the construction (1)–(5) a finite number of times, we have that the compact subset K_0' given by the intersection of all possible K' with $\Omega_0 \cup K$ is clearly a nonempty set and $\partial K_0' \cap \Omega_0$ is a compact set inside K. Therefore any trajectory of H_{f_1} starting at a point in $\Omega_0 \setminus K_0'$ will have both the α-limit and ω-limit sets on the part of the boundary of Ω which is inside K. So the continuous dependence argument can be applied once again to prove that $D = 0$. This concludes the proof that (3.1) is true.

Secondly if we denote $\mathcal{K}_\infty = J \setminus J'$ we show that given a trajectory δ of H_{f_1} then we have:

$$\omega(\delta) \subset \mathcal{K}_\infty \quad \text{or} \quad \alpha(\delta) \subset \mathcal{K}_\infty. \tag{3.2}$$

If neither of the above is true for some δ we take $j_1, j_2 \in J'$ such that $\omega(\delta) \cap K_{j_1} \neq \emptyset \neq \alpha(\delta) \cap K_{j_2}$. From Proposition 1.1 there are C^1 simple curves, γ_l, $l = 1, 2$, contained in Ω such that γ_1 contains K_{j_1} in its interior and K_{j_2} in its exterior, and γ_2 the other way around. Now take $K = \mathrm{Image}(\gamma_1) \cup \mathrm{Image}(\gamma_2)$, by arguing as in the proof that (3.1) with $\Delta_0 = \delta$ follows that for this K there is no K' satisfying the F-convexity condition for Ω. This proves (3.2).

Thirdly we prove that there exists $j_0 \in J$ such that

$$J' = \{j_0\} \quad \text{and} \quad \omega(\delta) \subset K_{j_0} \quad \text{or} \quad \alpha(\delta) \subset K_{j_0} \tag{3.3}$$

for all trajectory δ of H_{f_1}.

Suppose this is not the case. Let $j_1 \in J'$ and δ satisfying (3.1), it is easy to see that either $\omega(\delta) \subset K_{j_1}$ or $\alpha(\delta) \subset K_{j_1}$, because once we have a point of $\omega(\delta)$ (or $\alpha(\delta)$) in K_{j_1} then we cannot have another point of this set in another K_{j_2}, with $j_2 \in J \setminus \{j_1\}$, because by choosing

γ be a C^1 simple curve containing K_{j_1} in its interior and K_{j_2} in its exterior, as before taking $K = \text{Image}(\gamma)$ we get a contradiction with the hypothesis of F-convexity for Ω.

The next step is to improve (3.1). More precisely, (3.1) is true for any $j_1 \in J'$ and for some trajectory δ of H_{f_1}. Suppose that is not the case, then by Proposition 1.1 we take γ be a C^1 simple curve of Ω containing such K_{j_1} in its interior. If any trajectory of H_{f_1} passing through a point of γ has ω-limit and α-limit in the exterior of γ it is easy to see that the technique of proof of (3.1) works, so that we get a contradiction. So there is a trajectory of H_{f_1} passing through a point of γ having ω-limit or α-limit in the interior of γ, belonging to, say, K_{j_2} contained in the interior of γ, in particular $j_2 \in J'$. Now by Proposition 1.1 take γ' be a C^1 simple curve of Ω containing K_{j_1} in its interior and K_{j_2} in its interior. Separating K_{j_1} from K_{j_2}, keeping K_{j_1} in the interior of the curve, by interacting the argument above we get (3.1) is true also for K_{j_1}, which contradicts the hypothesis.

The proof of (3.3) is completed if we prove that $J' = \{j_0\}$. Suppose $j_1 \in J' \setminus \{j_0\}$, by Proposition 1.1 we take γ_i be a C^1 simple curve of Ω containing K_{j_i} in its interior and K_{j_j} in its exterior, where $i, j \in \{0, 1\}$ and $i \neq j$. From the connectivity of Ω take γ be a C^1 curve joining a point of $\text{Image}(\gamma_0)$ with a point of $\text{Image}(\gamma_1)$. And finally consider $K = \text{Image}(\gamma_0) \cup \text{Image}(\gamma_1) \cup \text{Image}(\gamma)$. For this K we get a contradiction with the hypothesis of F-convexity for Ω.

From (3.2) and (3.3) the proof of the theorem is completed.

Example. Let $\Omega = B_1(0) \setminus \{0\} =$ a punctured disk, and consider $F : \Omega \to \mathbb{R}^2$ defined by

$$F(x_1, x_2) = \bigl(x_1/r, \cos(r-1)x_2/r + \text{sen}(r-1)x_1/r\bigr) = \bigl(\cos(\theta), \text{sen}\bigl(\theta + (r-1)\bigr)\bigr).$$

It is easy to show that Ω is connected and Ω is F_i-convex for $i \in \{1, 2\}$, in particular we have that F is injective, but Ω is not simply connected. This example shows that Theorem 0.3 cannot be improved.

Proof of Corollary 0.1. Take f_i such component of F. We are going to show that such component satisfies the property of Definition 0.1. If this is not the case, we can apply Lemma 1.2 to V and Ω_1. Since V is bounded away from the origin, $D(x^2, y^2)$ goes to infinity as we take $x_1 \in \gamma((0, 1))$ and y^1 approaching a point of Σ_1, contradicting the conclusion of Lemma 1.2. This proves Corollary 0.1. $\square$

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References

[1] J.J. Duistermaat, L. Hörmander, Fourier integral operators II, Acta Math. 128 (1972) 183–269.

[2] M.K. Fort Jr, The complements of bounded, open connected subsets of Euclidean space, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 (6) (1961) 457–460.

[3] A. Gasull, J. Llibre, J. Sotomayor, Global asymptotic stability of differential equations in the plane, J. Differential Equations 91 (1991) 327–335.
[4] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vols. 1–4, Springer, Berlin.
[5] G. Meisters, Inverting polynomial maps on n-sphere by solving differential equations, in: Fink, Miller, Kliemann (Eds.), Delay and Differential Equations, World Scientific, Singapore, 1991, pp. 107–155.
[6] C. Olech, On the global stability of an autonomous system on the plane, Contrib. Differential Equations 1 (1963) 389–400.
[7] S. Pinchuk, A counterexample to the strong real Jacobian conjecture, Math. Z. 217 (1994) 1–4.
[8] R. Plastock, Homeomorphisms between Banach spaces, Trans. Amer. Math. Soc. 200 (1974) 169–183.